Elasticity model for blood vessel

V. Louvet [1],

Résumé

Keywords : elasticity, Strain Energy Function, arteries wall .

Table des matières

[1] I.C.J., Université Lyon 1, 43 bd du 11 novembre 1918, 69622 Villeurbanne Cedex, France. e-mail : louvet@math.univ-lyon1.fr

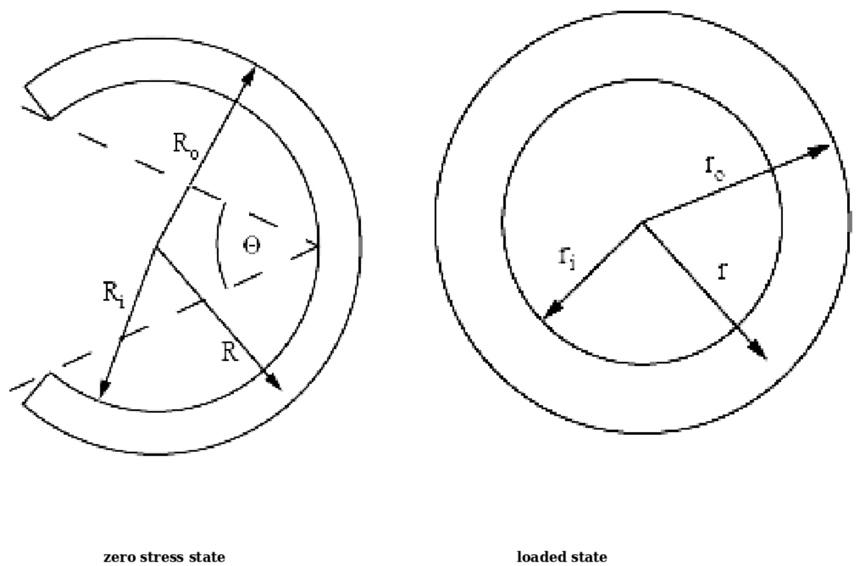
1 Introduction

The aim of this paper is to provide a study of different models of the mechanical non-linear stress-strain behaviour of arteries.

2 Generic Model for the SEF

We use a cylindrical coordinate system with the pricipal directions $\mathbf{e}_\theta$ (circumferential), $\mathbf{e}_z$ (axial) an $\mathbf{e}_r$ (radial).

FIG. 1 – Parameters characterizing the loaded state and the zero-stress state geometry of an artery. L, zero-stress state axial length ; l, loaded state axial length ; R_i, zero-stress state inner radius ; R_o, zero-stress state outer radius ; R, zero-stress state radius of wall point ;Θ, opening angle ; r_i, loaded state inner radius ; r_o, loaded state outer radius ; r, loaded state radius of wall point.



The SEF is expressed in a cylindrical coordinate system in terms of local Green strains (E_θ, E_r, E_z) or stretch ratios $(\lambda_\theta, \lambda_r, \lambda_z)$ which are related to each other via the relation

$$\lambda_k = \sqrt{2E_k + 1}\; k = \theta, z, r \tag{1}$$

Each stretch ratio can be expressed with the parameters of the different states (2) as follows :

$$\left\{ \begin{array}{lcl} \lambda_\theta & = & \dfrac{\pi}{\pi - \Theta} \dfrac{r}{R} \\ \\ \lambda_z & = & \dfrac{l}{L} \\ \\ \lambda_r & = & \dfrac{\partial r}{\partial R} \end{array} \right. \tag{2}$$

It is assumed that the arterial wall material is incompressible, so that

$$\lambda_\theta \lambda_z \lambda_r = 1$$

Solving the system (2) under the boundary condition that the outer radius in the Zero Stress State R_o should be mapped to the outer radius in the stretched and pressurized state, $r_o = r(R_o)$, we find :

$$r(R) = \sqrt{r_o^2 - (R_o^2 - R^2) \frac{\pi - \Theta}{\lambda_z \Theta}} \tag{3}$$

In the absence of body forces the equilibrium equations are

$$\text{div } \sigma = 0 \tag{4}$$

where σ is the Cauchy Stress tensor and div denotes the spatial divergence of the spatial tensor field. Note that in cylindrical polar coordinates (r, θ, z), because of the geometrical and constitutive symmetry, the only non-trivial component of (4) is the following, with boundary conditions related to p_o and p_i, the external and internal pressure :

$$\left\{ \begin{array}{ccc} \dfrac{d\sigma_r}{dr} + \dfrac{\sigma_r - \sigma_\theta}{r} & = & 0 \qquad \text{with } \; r_i \le r \le r_0 \\ \sigma_r(r_0) & = & -p_0 \\ \sigma_r(r_i) & = & -p_i \end{array} \right. \tag{5}$$

The internal pressure is then obtained in the form :

$$p_i = \int_{r_i}^{r_o} (\sigma_\theta - \sigma_r) \frac{dr}{r} \tag{6}$$

For axisymmetric loading, the stress-strain relationships can be derived by means of a strain energy function $\psi(E_r, E_\theta, E_z)$. In this case, the radial, circumferential and longitudinal stresses are given by :

$$\left\{\begin{array}{lcl} \sigma_r & = & \lambda_r^2 \dfrac{\partial \psi}{\partial E_r} + q \\ \\ \sigma_\theta & = & \lambda_\theta^2 \dfrac{\partial \psi}{\partial E_\theta} + q \\ \\ \sigma_z & = & \lambda_z^2 \dfrac{\partial \psi}{\partial E_z} + q \end{array}\right. \tag{7}$$

Many SEF have been proposed for constitutive models of arteries wall. In the next sections, some of this models are detailed and compute.

In order to study the different models, we are interesting in the different following variables :

- the diameter of arterial wall : $D = 2 \times r_i$,
- the compliance : $C = \dfrac{\partial D(P)}{\partial P}$ where P is the intraluminal pressure,
- the distensibility : $D_d = \dfrac{\partial D(P)}{D \partial P}$,
- the thickness : $h = r_o - r_i$,
- the young's modulus : $E = \dfrac{\partial P \times D}{\partial D \times h}$,
- the pulse wave velocity compute : $PWV = \sqrt{\dfrac{Eh}{2\rho r_i}}$, where ρ is the blood density ($\rho = 1.05$).

Some characteristics of the Cauchy-Green deformation tensor :

$$\mathbf{C} = 2\mathbf{E} + \mathbf{1} \tag{8}$$

will be although usefull :

- the first invariant : $I_1 = tr(\mathbf{C}) = \lambda_\theta^{\mathbf{2}} + \lambda_{\mathbf{r}}^{\mathbf{2}} + \lambda_{\mathbf{z}}^{\mathbf{2}}$
- the invariant with respect to a vector $\mathbf{v}_\theta$ definig a fiber orientation defined in tensor notation by : $I_4 = \mathbf{v}_\alpha \mathbf{C} \mathbf{v}_\alpha$.

  In absence of torsion and if $\mathbf{v}_\alpha$ is in the $\mathbf{e}_\theta - \mathbf{e}_z$ plane at an angle α to the circumferential direction $\mathbf{e}_\theta$, I_4 takes on the following form :

$$I_4 = \lambda_\theta^2 \cos^2 \alpha + \lambda_z^2 \sin^2 \alpha$$

.

– Similarly, we can define the invariant $I_{4'}$ in respect to a vector $\mathbf{v}_{\alpha'}$ having a second fiber direction defined by an angle α'.

3 Choung and Fung model

3.1 mathematical model

The most widely known SEF for arteries is probably the exponential-polynomial SEF of Chuong and Fung [1]. It displays both non-linearity and anisotropy :

$$\psi(E_r, E\theta, E_z) = \frac{c}{2}(\exp^{(b_1 E_\theta^2 + b_2 E_z^2 + b_3 E_r^2 + b_4 E_\theta E_z + b_5 E_z E_r + b_6 E_r E_\theta)} - 1) \qquad (9)$$

where c is an elastic constant and b_1 to b_6 are parameters describing the contributionof the principal strains. This model is purely phenomenological.

3.2 Computational results

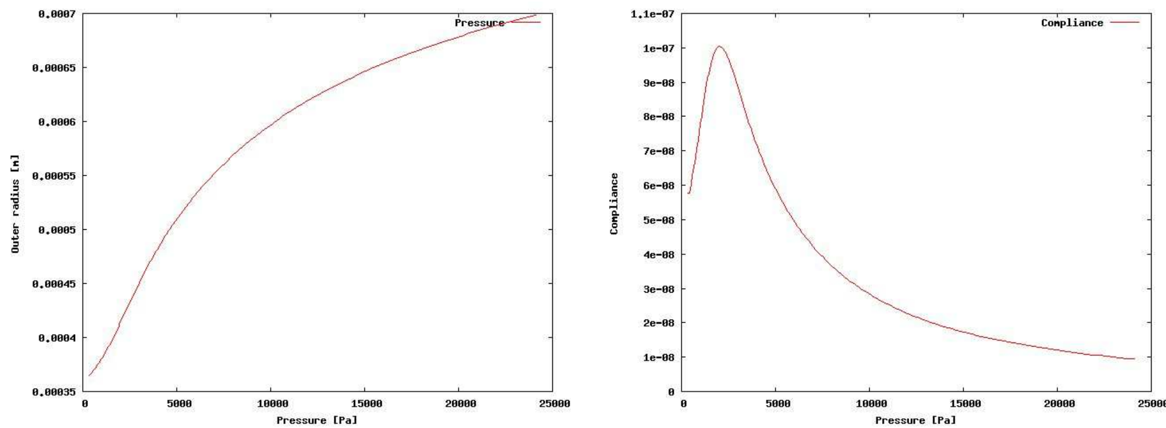
Using (9) and (6) it is easy to compute the model associate to the SEF of Choung and Fung.

The differents parameters used are obtained by fitted experimental datas from rat carotid artery (see [3]) :

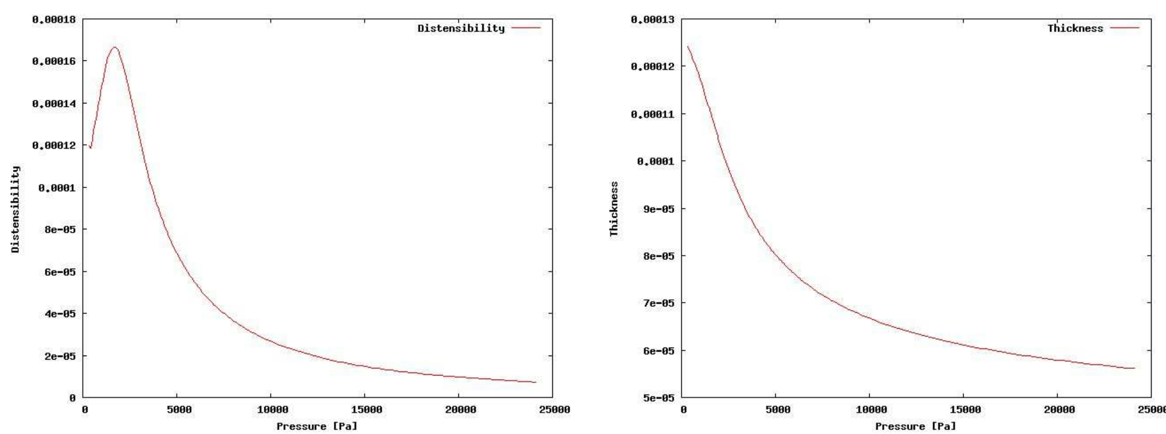
$$\left\{ \begin{array}{rcl} c & = & 222.24\ kPa \\ b_1 & = & 0.2634 \\ b_2 & = & 0.2135 \\ b_3 & = & 0.3759 \\ b_4 & = & 0.1832 \\ b_5 & = & 0.1951 \\ b_6 & = & 0. \end{array} \right. \qquad (10)$$

We consider the parameters of the artery as follow :

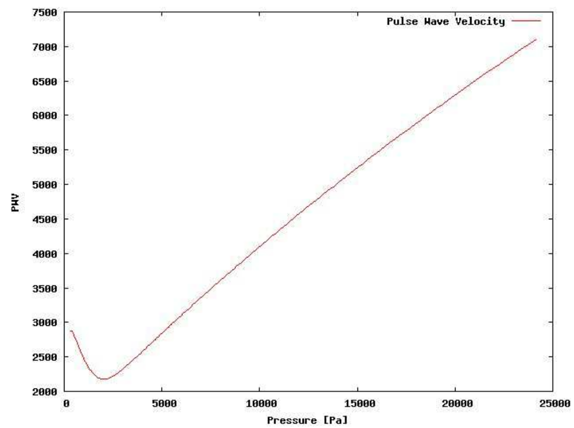
$$
\begin{cases}
L & = & 1. \\
l & = & 1.3 \\
R_o & = & 0.5845E-03 \\
R_i & = & 0.4415E-03 \\
\Theta & = & 60.
\end{cases}
\tag{11}
$$



TAB. 1 – Curves of pressure and compliance for Choung model.



TAB. 2 – Curves of distensibility and thickness for Choung model.



TAB. 3 – Curves of pulse wave velocity for Choung model.

4 Holzapfel model

4.1 mathematical model

Holzapfel and Gasser [2] develop a constitutive law for the description of the passive mechanical response of arterial tissue. This SEF is split into two different parts, one associated with isotropic deformations et the other associated with anisotropic deformations :

$$\psi = \psi_{iso} + \psi_{aniso} \tag{12}$$

where

$$\psi_{iso} = \frac{c}{2}(I_1 - 3) \tag{13}$$

and

$$\psi_{aniso} = \frac{k_1}{k_2}\left[\frac{1}{2}(\exp^{k_2(I_4-1)^2} -1) + \frac{1}{2}(\exp^{k_2(I_{4'}-1)^2}\right] \tag{14}$$

We consider for I_4 and $I_{4'}$ invariants, $\alpha = \alpha'$ and then $I_4 = I_{4'}$.

In the Holzapfel model, c and k_1 are positive material constants with the dimension of the stress and k_2 is a dimensionless parameter.

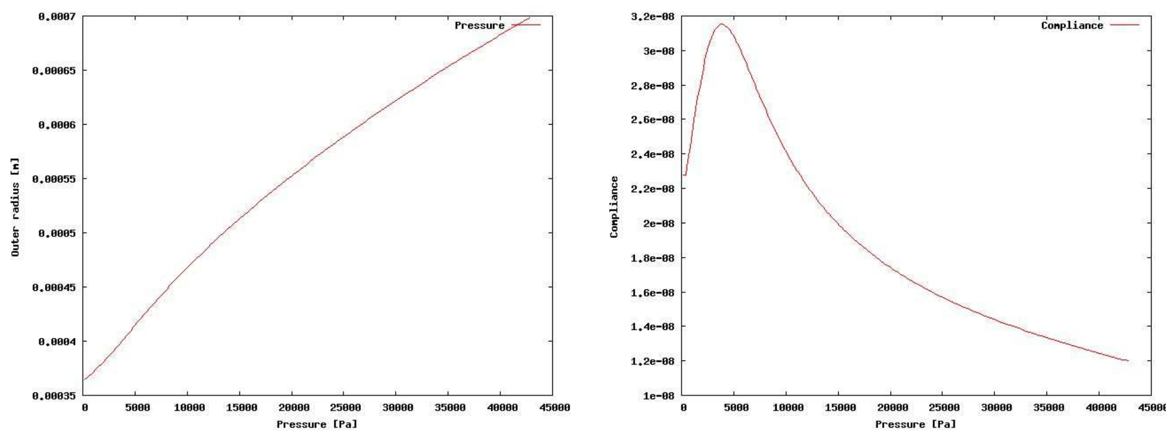
4.2 Computational results

The differents parameters used are obtained by fitted experimental datas from rat carotid artery (see [3]) :

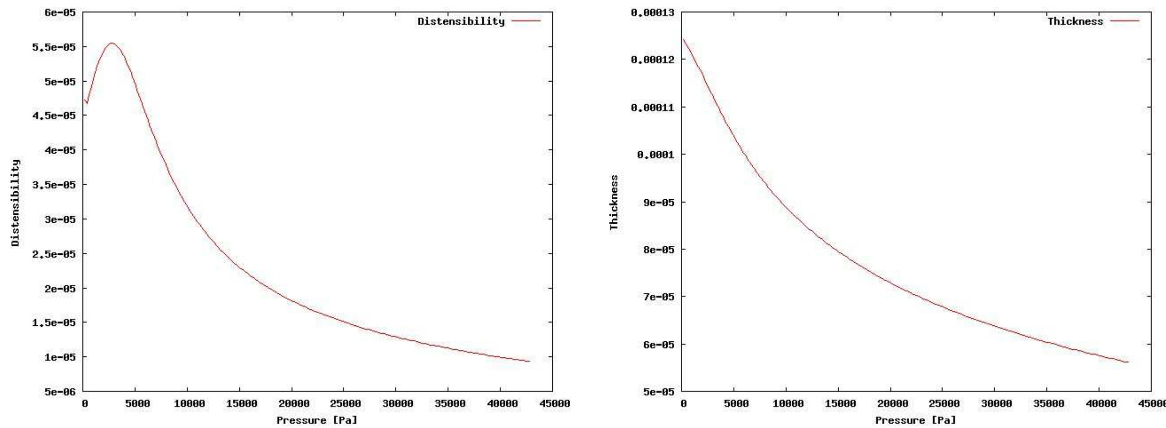
$$\left\{\begin{array}{rcl} c & = & 44.24\,kPa \\ k_1 & = & 0.206\,kPa \\ k_2 & = & 1.465 \\ \alpha & = & 39.76^\circ \end{array}\right. \tag{15}$$

We consider the parameters of the artery as follow :

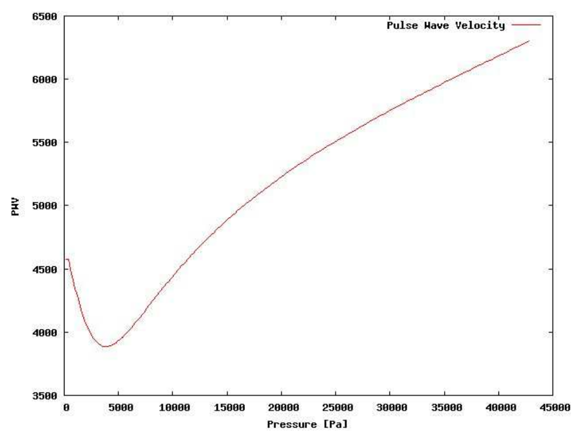
$$\left\{\begin{array}{rcl} L & = & 1. \\ l & = & 1.3 \\ R_o & = & 0.5845E-03 \\ R_i & = & 0.4415E-03 \\ \Theta & = & 60. \end{array}\right. \tag{16}$$



TAB. 4 – Curves of pressure and compliance for Holzapfel model.



TAB. 5 – Curves of distensibility and thickness for Holzapfel model.



TAB. 6 – Curves of pulse wave velocity for Holzapfel model.

5 Zulliger model

5.1 mathematical model

The general approach is similar to the above model concerning the separation of the SEF into isotropic and anisotropic parts. In this case, the isotropic ψ_{iso} is assumed to represent the elastin alone, using the total wall cross-section area which is composed of elastin f_{elast} :

$$\psi_{iso} = f_{elast}\psi_{elast} \tag{17}$$

where

$$\psi_{elast} = c_{elast}(I_1 - 3)^{3/2} \tag{18}$$

In a similar way, the anisotropic component is assumed to represent only the load bearing collagen fibers :

$$\psi_{aniso} = f_{coll}\psi_{coll} \tag{19}$$

where f_coll is the corresponding area fraction of collagen.

To describe the ensemble of circumferentially oriented collagen fibers, it is assumed that the engagement of the collagen fibers when stretched is distributed in some statistical manner, with the log-logistic probability distribution function :

$$\rho_{fiber}(E) = \begin{cases} 0 & \text{for } E \leq E_0 \\ \dfrac{k}{b}\dfrac{\left(\dfrac{E-E_0}{b}\right)^{k-1}}{\left[1+\left(\dfrac{E-E_0}{b}\right)^{k}\right]^2} & \text{for } E > E_0 \end{cases} \tag{20}$$

where $b > 0$ is a scaling parameter and $k > 0$ defines the shape of the distribution. To prevent any collagen fibers exerting a force when the tissue is in its ZSS, $E_0 = 0$. An individual collagen fiber's SEF is described by :

$$\psi_{fiber}(E') = \begin{cases} 0 \text{ for } E' \leq 0 \\ c_{coll}\dfrac{1}{2}E'^2 \text{ for } E' > 0 \end{cases} \tag{21}$$

where c_{coll} is the elastic constant associated with the collagen, and E' the local strain in direction of the fiber.

We then have the whole contribution for the collagen :

$$\begin{aligned} \psi_{coll}(E_\theta) = \psi_{fiber} \star \rho_{fiber} &= \int_{-\infty}^{\infty} \psi_{fiber}(E').\rho_{fiber}(E_\theta - E')dE' \\ &= \int_{0}^{E_\theta} \psi_{fiber}(E').\rho_{fiber}(E_\theta - E')dE' \end{aligned} \tag{22}$$

The passive components of the arterial wall (elastin and collagen) are treated together under the form :

$$\psi_{passive} = f_{elast}\psi_{elast} + f_{coll}\psi_{coll} \tag{23}$$

In contrast to previous models that included the effects of smooth muscle contraction through generation of an active stress, in this study, the model proposed by Zulliger and all consider the vascular muscle as a structural element whose contribution to load bearing is modulated by the contraction [4].

To extend the mechanical description of the arterial wall to include the effects of VSM tone, Zulliger and all propose adding an additional term to the above-mentioned SEF (24) :

$$\psi = \psi_{passive} + S_1 S_2 f_{VSM}\psi_{VSM} \tag{24}$$

where f_{VSM} is the cross-sectional area fratiocn of VSM, and ψ_{VSM} is a SEF describing the VSM when maximally contracted.

S_1 is a nondimensional function describing the level of VSM tone :

$$S_2 = \begin{cases} 1 & 0.680 < \lambda_\theta^{VSM}/\lambda_{pre} < 1.505 \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

where λ_θ^{VSM} is the circumferebntial stretch of the VSM when the artery is in its maximally contracted state :

$$\lambda_\theta^{VSM} = \lambda_\theta \lambda_{pre} \tag{26}$$

and $1/\lambda_{pre}$ can be understood as length of an isolated VSM cell under the isotonic maximal contraction relative to the length of the same VSM cell when fully relaxed and embedded in the extracellular matrix.

S_2 incorporates the range of stretch at which the VSM develops maximal force under isometric contraction :

$$S_1 = \begin{cases} 0 & \text{fully relaxed} \\ 1 & \text{maximally contracted} \\ S_{basal} + (1 - S_{basal})\dfrac{1}{2}\left[1 + Erf\left(\dfrac{Q-\mu}{\sqrt{2}\sigma}\right)\right] & \text{normal tone} \end{cases} \tag{27}$$

Q is a function of the VSM deformation :

$$Q = \alpha_\theta E_\theta + \alpha_z E_z + \alpha_r E_r + 3 \tag{28}$$

where α_θ, α_z and α_r describe the sensitivity of the VSM to deformations in the corresponding directions. S_{basal} represents the VSM basal tone contraction. It is assume a Gaussian distribution of the VSM cell activation level as a function of deformation Q.The Gaussian distribution is characterized by the critical engagement deformation, μ, and a half-width, σ, and is represented by the error function Erf.

When maximally contracted, the VSM cells' contribution to the total SEF is assumed to be described by the following relationship

$$\psi_{VSM} = c_{VSM}\left[\lambda_\theta^{VSM} - \log(\lambda_\theta^{VSM}) - 1\right] \tag{29}$$

C_{VSM} takes the role of an elastic modulus.

5.2 Computational results

5.3 Parameters

We consider the parameters of the artery as follow :

$$\left\{\begin{array}{lcc} L & = & 1. \\ l & = & 1. \\ R_o & = & 0.5845E-03 \\ R_i & = & 0.4415E-03 \\ \Theta & = & 60. \end{array}\right. \tag{30}$$

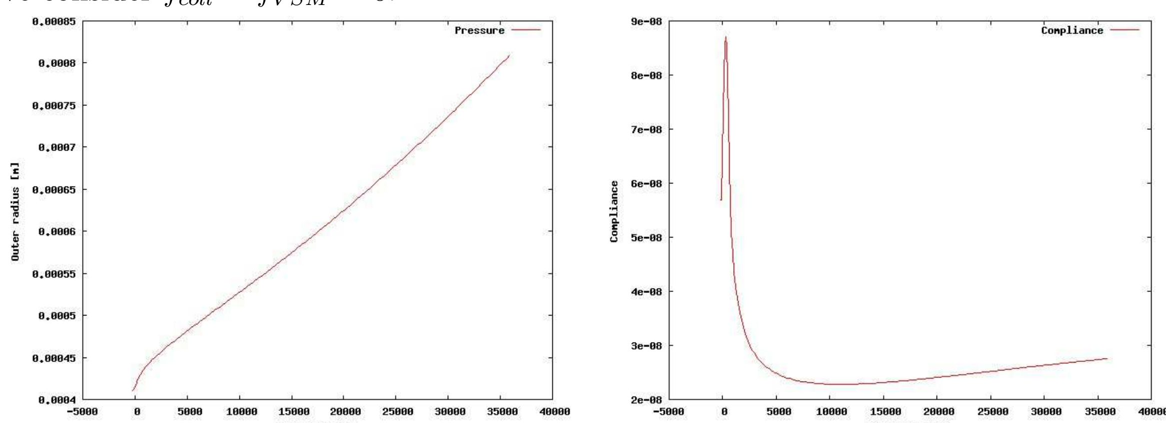
The differents parameters used are obtained by fitted experimental datas from left common carotid (LCC) artery of E+/+ mouse in the normatensive case :

$$
\left\{
\begin{array}{ccc}
S_{basal} & = & 0.052 \\
\mu & = & 4.81 \\
\lambda_{pre} & = & 1.83 \\
\sigma & = & 0.133 \\
\alpha_\theta & = & \alpha_r = \alpha_z = 2. \\
f_{elast} & = & 0.306 \\
c_{elast} & = & 52.10^3 \\
f_{coll} & = & 0.203 \\
c_{coll} & = & 200.10^6 \\
k & = & 21. \\
b & = & 1.6 \\
f_{VSM} & = & 0.491 \\
c_{VSM} & = & 73.2.10^3
\end{array}
\right. \tag{31}
$$

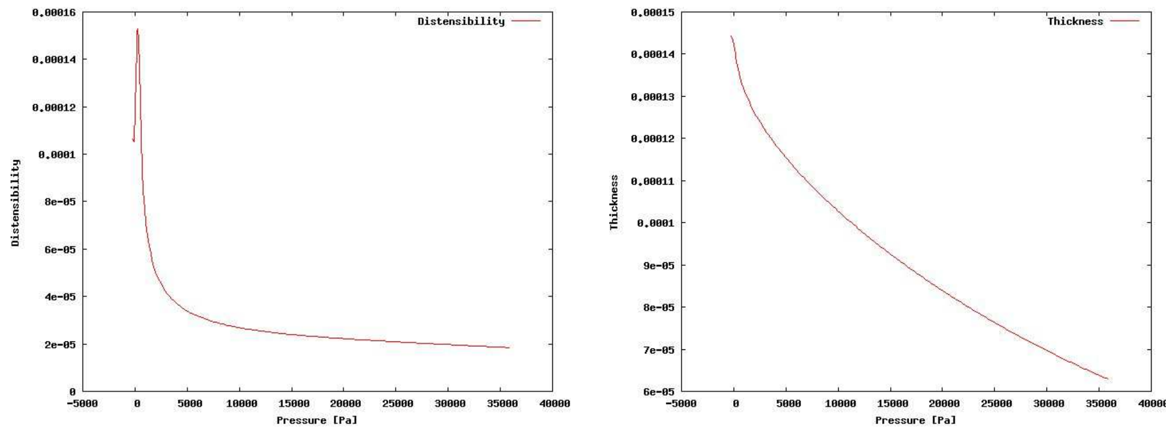
To understand the behavior of each component (elastin, collagen and VSM), it can be useful to cancel the action of the other.

5.3.1 Action of the elastin

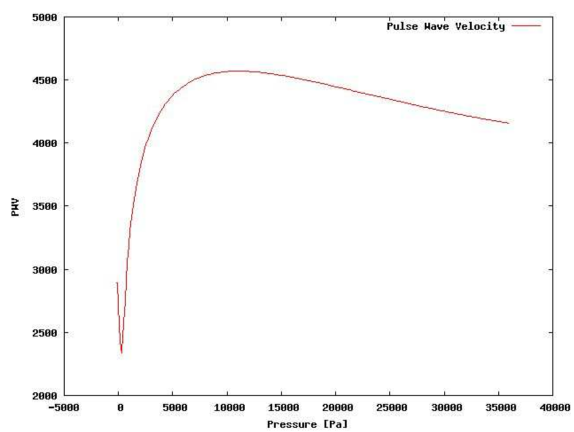
We consider $f_{coll} = f_{VSM} = 0$.



TAB. 7 – Curves of pressure and compliance for Zulliger model with only action of elastin .



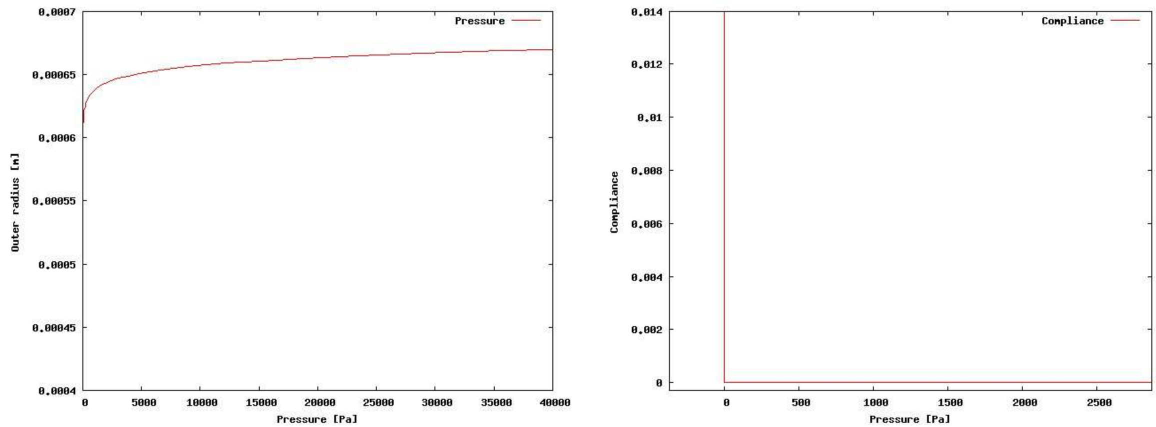
TAB. 8 – Curves of distensibility and thickness for Zulliger model with only action of elastin .



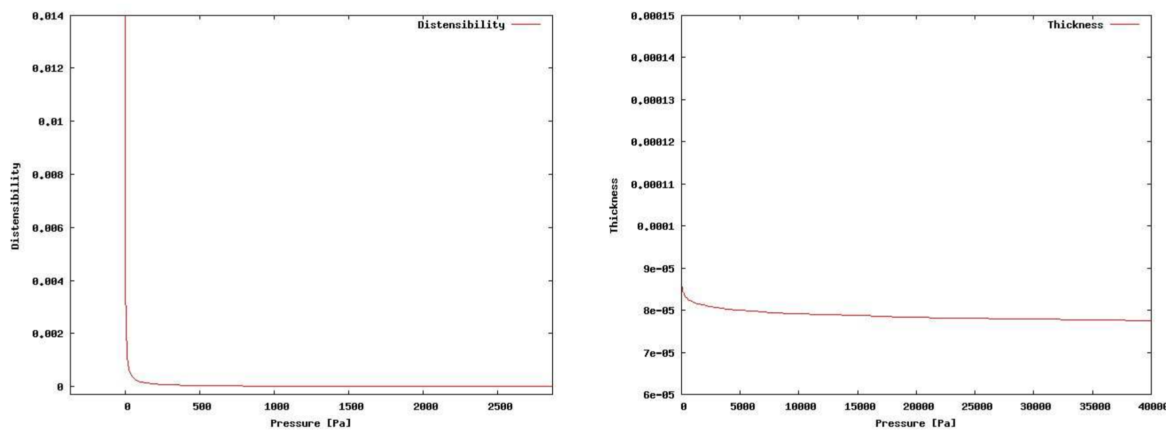
TAB. 9 – Curves of pulse wave velocity for Zulliger model with only action of elastin .

5.3.2 Action of the collagen

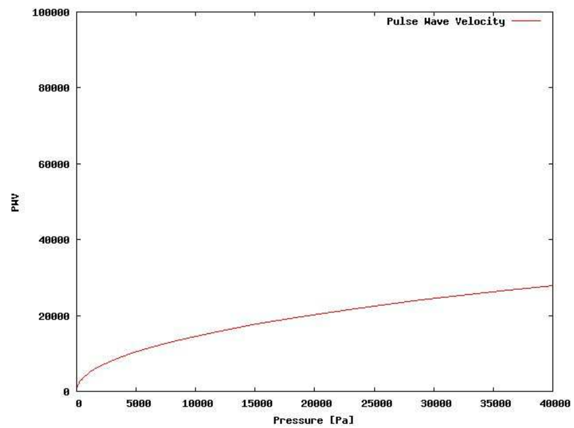
We consider $f_{elast} = f_{VSM} = 0$.



Tab. 10 – Curves of pressure and compliance for Zulliger model with only action of collagen.



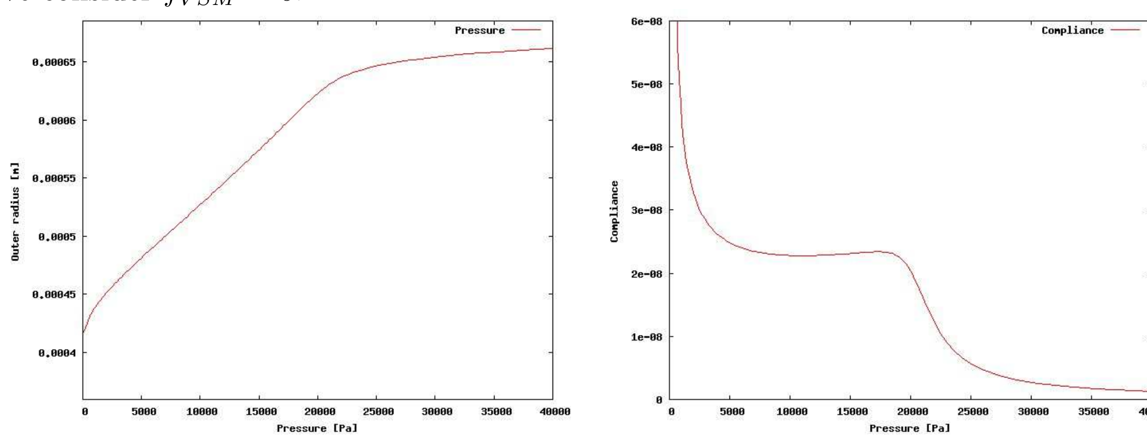
Tab. 11 – Curves of distensibility and thickness for Zulliger model with only action of collagen.



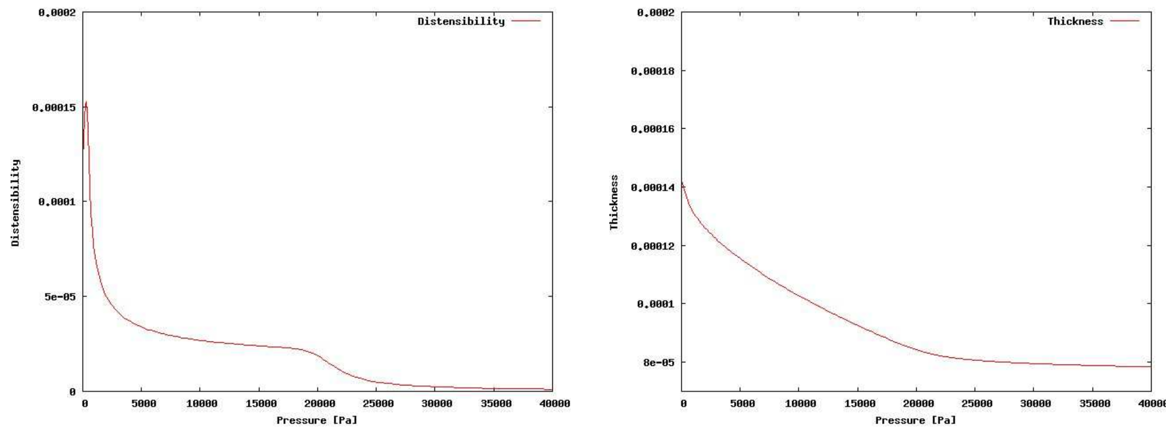
TAB. 12 – Curves of pulse wave velocity for Zulliger model with only action of collagen.

5.3.3 Action of the elastin and the collagen

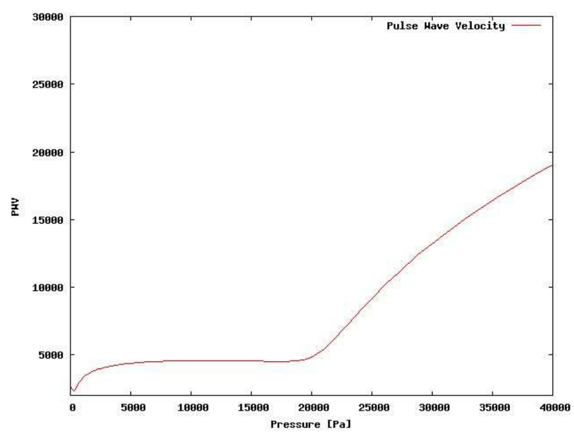
We consider $f_{VSM} = 0$.



TAB. 13 – Curves of pressure and compliance for Zulliger model with only action of elastin and collagen.



TAB. 14 – Curves of distensibility and thickness for Zulliger model with only action of elastin and collagen.

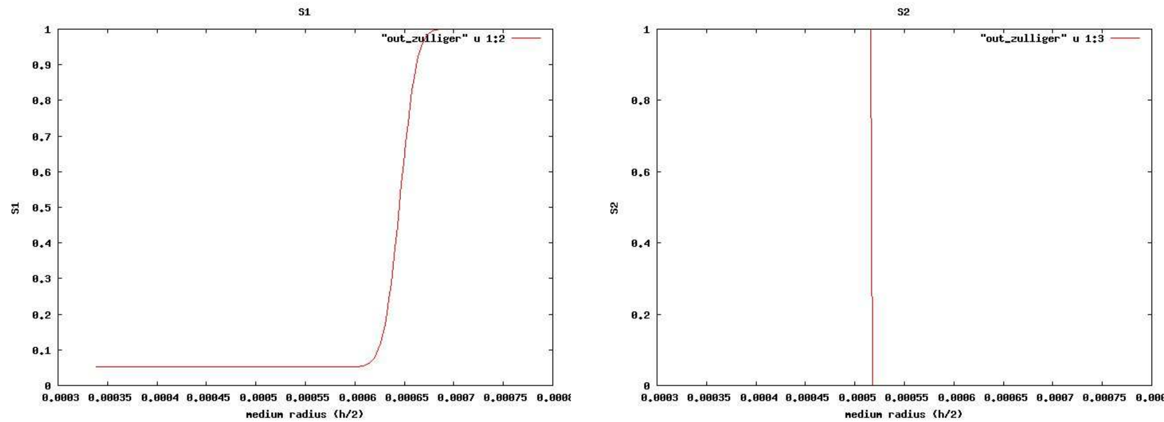


TAB. 15 – Curves of pulse wave velocity for Zulliger model with only action of elastin and collagen.

5.3.4 Action of the VSM

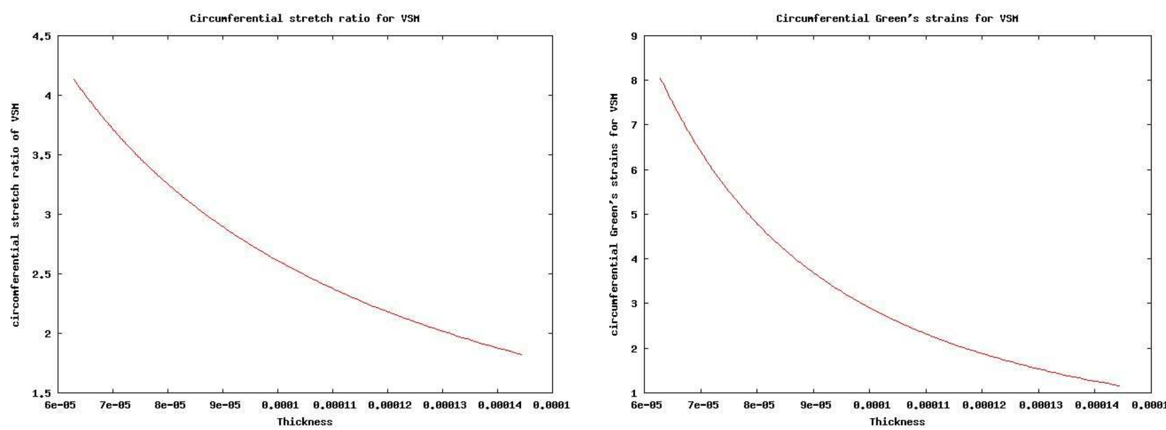
We consider $f_{elast} = f_{coll} = 0$.

First of all, we study the behavior of the parameters taking place in the model in function of the stretch ratios.

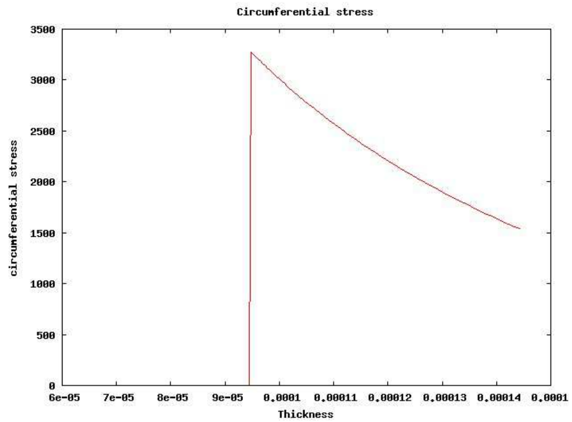


TAB. 16 – Curves of $S1$ and $S2$ parameters for Zulliger model with only action of VSM.

We are now interesting of each components of the circumferential stress.

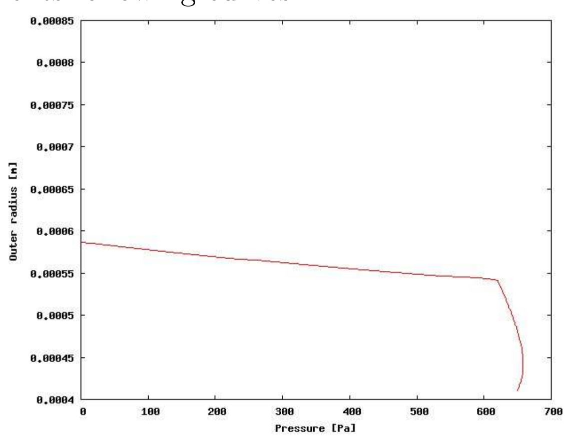


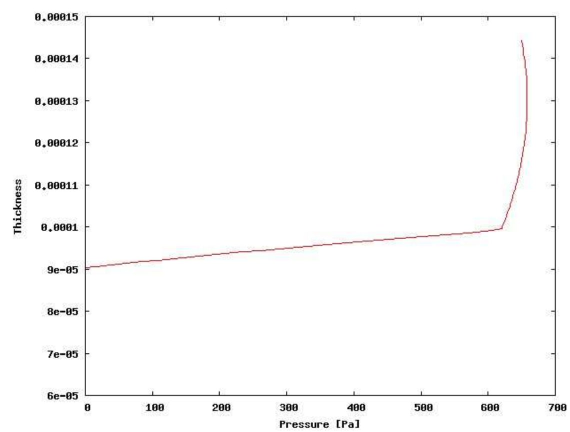
TAB. 17 – Curves of λ_θ^{VSM} and E_θ^{VSM} for Zulliger model with only action of VSM.



TAB. 18 – Curves of the circumferential stress σ_θ for Zulliger model with only action of VSM.

Concerning the same variables studies for the other model, we have the differents following curves.

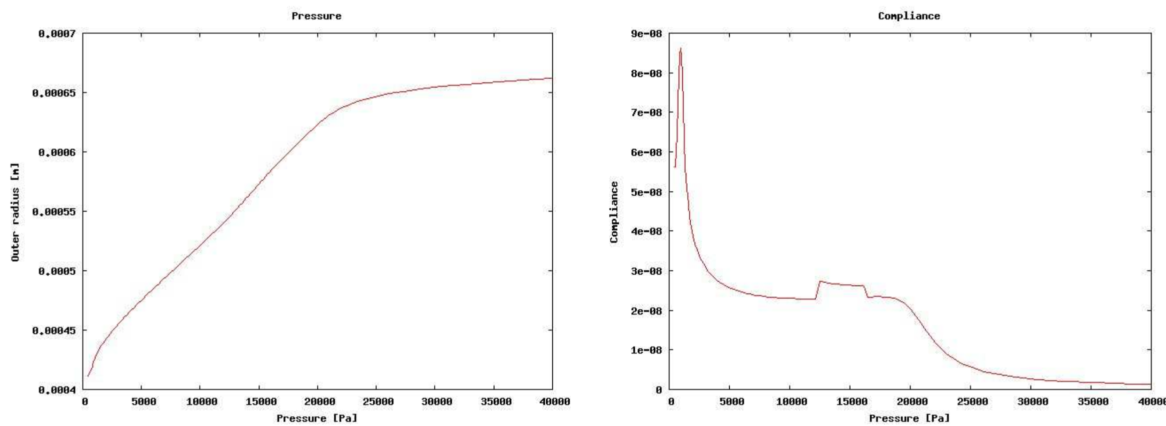




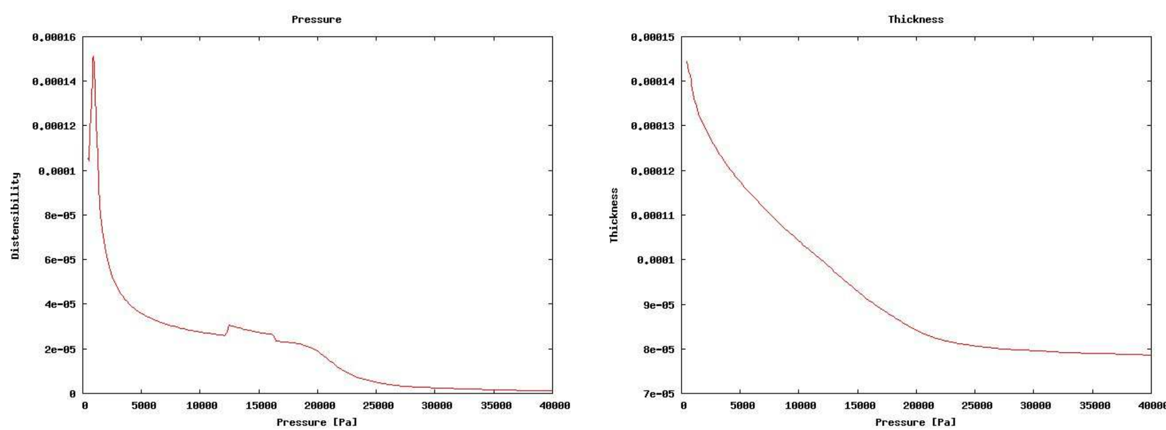
TAB. 19 – Curves of pressure and thickness for Zulliger model with only action of VSM.

The curves of distensibility, compliance and pulse wave velocity have no interest due to the shape of the pressure curve.

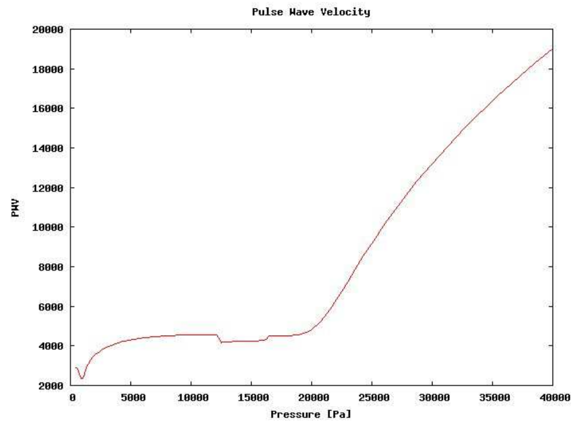
5.3.5 Action of all the components



TAB. 20 – Curves of pressure and compliance for Zulliger model.

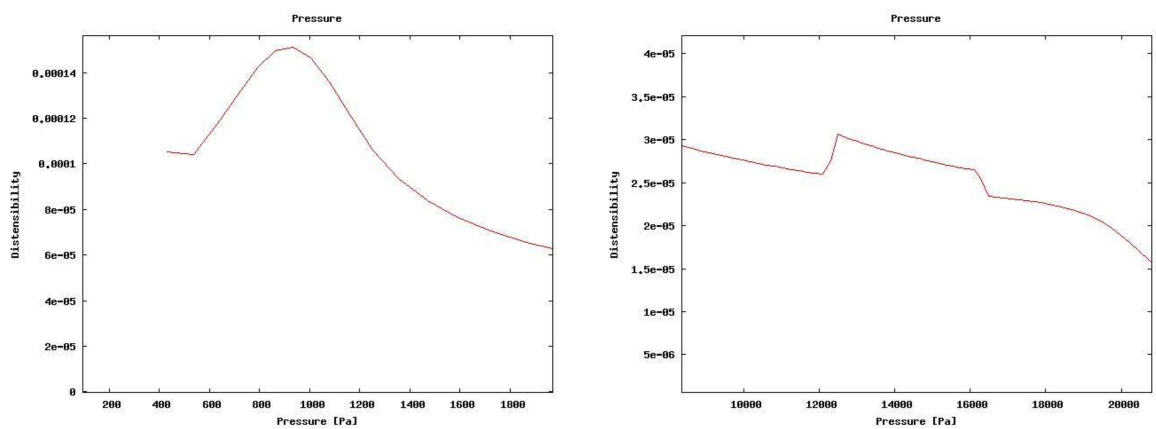


TAB. 21 – Curves of distensibility and thickness for Zulliger model.

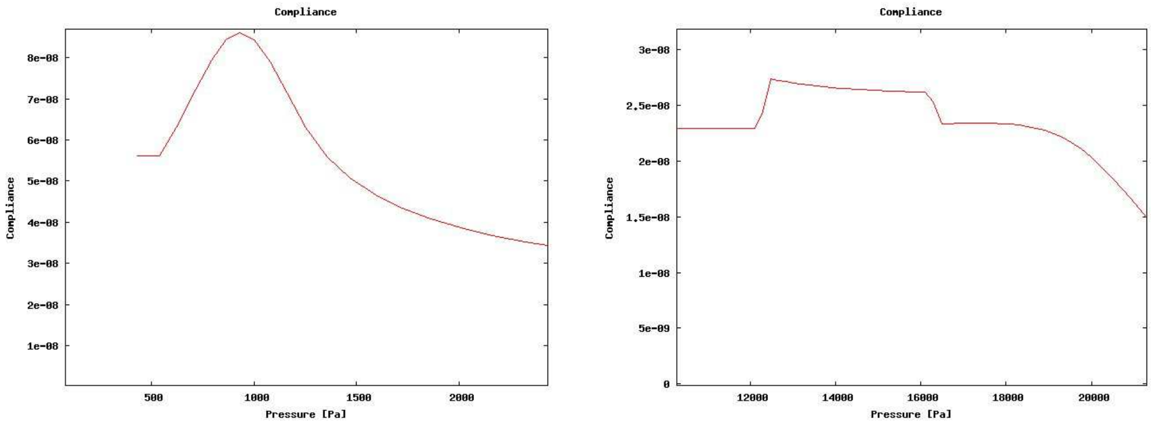


TAB. 22 – Curves of pulse wave velocity for Zulliger model.

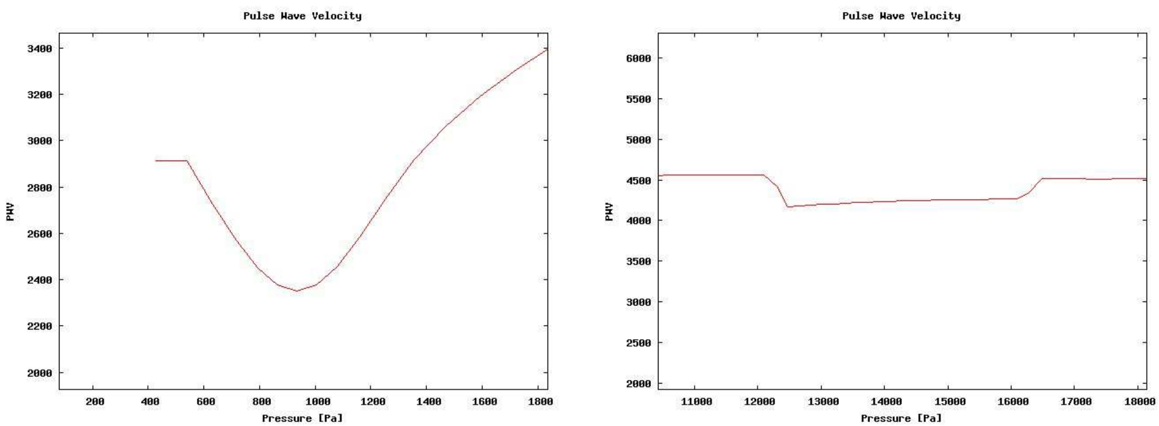
Some of this curves presents a curious behavior for some value of pressure. The next ones show some particular parts of the pulse wave velocity, distensibility and compliance curves.



TAB. 23 – Zoom on the distenbility curves for Zulliger model.



TAB. 24 – Zoom on the compliance curves for Zulliger model.



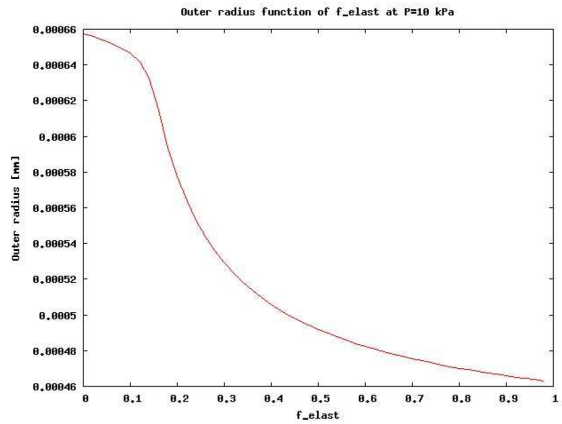
TAB. 25 – Zoom on the pulse wave velocity curves for Zulliger model.

5.3.6 Variation of the total wall cross-section area composed by elastin

We consider $f_{VSM} = 0$ because of the above curves and we study the effect of the variation of elastin on the outer radius with a fixed pressure.

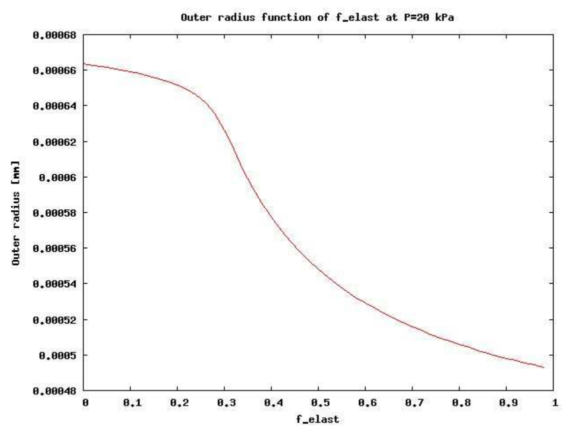
We make f_elast vary from 0 to 1. We use the same parameters as above.

Pressure = 10 kPa



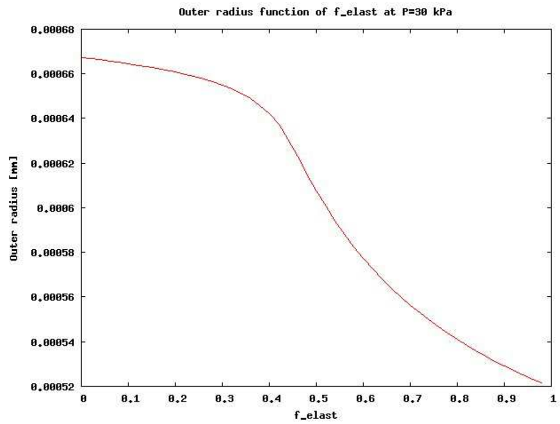
TAB. 26 – Curves of outer radius function of f_{elast} for $P = 10kPa$.

Pressure = 20 kPa



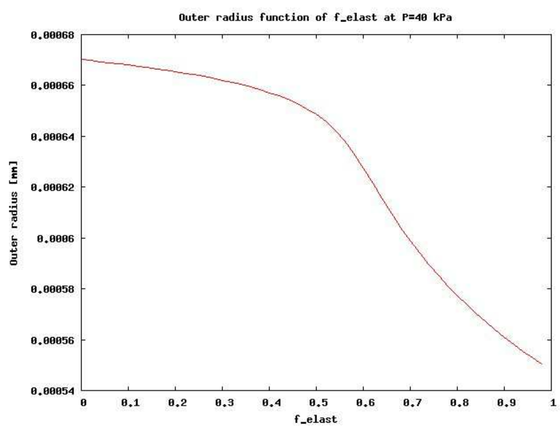
TAB. 27 – Curves of outer radius function of f_{elast} for $P = 20kPa$.

Pressure = 30 kPa



TAB. 28 – Curves of outer radius function of f_{elast} for $P = 30kPa$.

Pressure = 40 kPa



TAB. 29 – Curves of outer radius function of f_{elast} for $P = 40kPa$.

This work was realized in collaboration with L. HUA [2], B. KASSAI [3] and E. GRENIER [4].

[2]Service de Pharmacologie Clinique, Université Lyon 1. e-mail : laure_hl@hotmail.com

[3]Service de Pharmacologie Clinique, Université Lyon 1. e-mail : bk@upcl.univ-lyon1.fr

[4]U.M.P.A., École Normale Supérieure de Lyon, 46, allée d'Italie, 69364 Lyon Cedex 07, France. e-mail : egrenier@umpa.ens-lyon.fr

Références

[1] Chuong CJ and Fung YC. *Three-dimensional stress distribution in arteries*, J Biomech Eng, vol 105, pp 268-274, 1983.

[2] Holzapfel GA and Gasser TC. *A new constitutive framwork for arterial wall mechanics and a comparative study of material models*, J of Elasticity, vol 61, pp 1-48, 2000.

[3] Zulliger MA, Fridez P, Hayashi K, Stergiopulos N. *A strain energy function for arteries accounting for wall composition and structure*, J of Biomechanics, vol 37, pp 989-1000, 2004.

[4] Zulliger MA, Rachev A, Stergiopulos N. *A constitutive formulation of arterial mechanics including vascular smooth muscle tone*, Am J Physil Heart Circ Physiol, vol 287, pp 1335-1343, 2004.